

GENERALIZED QUADRATIC FUNCTIONAL EQUATION WITH SEVERAL VARIABLES AND ITS HYERS-ULAM STABILITY

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ABSTRACT. In this paper, we introduce a generalized quadratic functional equation with several variables and then investigate its generalized Hyers-Ulam stability in normed spaces.

1. Introduction

In 1940, S. M. Ulam [14] suggested the following question associated with the stability of group homomorphisms: Let G be a group and let G' be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : G \rightarrow G'$ satisfies the inequality

$$d(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G$, then there exists a homomorphism $F : G \rightarrow G'$ with $d(f(x), F(x)) < \varepsilon$ for all $x \in G$?

The question of Ulam was first solved by D. H. Hyers [5] for approximate additive mappings between Banach spaces. In 1978 T. M. Rassias [9] and in 1991 Z. Gajda [3] provided a generalized Hyers-Ulam stability for the unbounded Cauchy difference controlled by a sum of unbounded function $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$, while $0 < p < 1$ and $p > 1$, respectively. In 1984, J. M. Rassias [10] gave a similar stability for the unbounded Cauchy difference controlled by a product of unbounded function $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon\|x\|^p\|y\|^q, p+q \neq 1$. In 1994, Găvruta [4] established a generalized Hyers-Ulam stability by replacing

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the bound of Cauchy difference controlled by a general control function like $\|f(x+y) - f(x) - f(y)\| \leq \varphi(x, y)$ with $\sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i y)}{2^i} < \infty$.

The following functional equation

$$f(x+y) + f(x-y) = 2[f(x) + f(y)]$$

is called a quadratic functional equation which may be originated from the important parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2[\|x\|^2 + \|y\|^2]$$

in inner product spaces. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping.

In 1983, The Hyers–Ulam stability problem for the quadratic functional equation was first proved by F. Skof [13] for a mapping $f : X \rightarrow Y$, where X is a normed space and Y is a Banach space. In 1992, S. Czerwik [2] demonstrated the Hyers–Ulam stability of the quadratic functional equation with the sum of powers of norms in the sense of T. M. Rassias approach using direct method. In the same year, J. M. Rassias [11] certified the Hyers–Ulam stability of the quadratic functional equation with the product of powers of norms using direct method. In 1995, C. Borelli and G. L. Forti [1] have verified the generalized Hyers–Ulam stability theorem of the quadratic functional equation. In 2003, Radu [12] proposed a new method, the fixed point method which is based on the fixed point alternative theorem to investigate the stability of functional equation. In 2008, S. -M. Jung and Z. -H. Lee [7] solved the stability of quadratic functional equation by using the fixed point approach.

Recently, A. Zivari-Kazempour and M. Eshaghi Gordji [15] have determined the general solution of the quadratic functional equation

$$\begin{aligned} f(x+2y) + f(y+2z) + f(z+2x) \\ = 2f(x+y+z) + 3[f(x) + f(y) + f(z)], \end{aligned}$$

and then have investigated its generalized Hyers–Ulam stability. Motivated from this quadratic functional equation, we consider a generalized functional equation

(1.1)

$$\begin{aligned} \sum_{i=1}^n \sum_{1 \leq i_1 \leq \dots \leq i_n \leq n} f\left(kx_i + \sum_{j=1, j_j \neq i}^{n-k} x_{i_j}\right) \\ =_{n-2} C_{n-k-1} \times (n+k-1) f\left(\sum_{i=1}^n x_i\right) +_{n-2} C_{n-k-1} \times \frac{nk(k-1)}{n-k} \sum_{i=1}^n f(x_i), \end{aligned}$$

where n, k are fixed positive integers with $n \geq 3$ and $2 \leq k \leq n - 1$.

In this article, we establish generalized Hyers–Ulam stability of the functional equation (1.1) by using the direct method and the fixed point method in normed spaces.

2. The generalized Hyers-Ulam stability of equation (1.1)

We begin with the general solution of the functional equation (1.1).

LEMMA 2.1. *Let X and Y be vector spaces and let k be a fixed positive integer with $k \geq 2$. If a mapping $f : X \rightarrow Y$ satisfies the functional equation*

$$(2.1) \quad f(kx + y) + f(x + ky) = 2kf(x + y) + (k - 1)^2[f(x) + f(y)]$$

if and only if it satisfies the quadratic functional equation

$$(2.2) \quad f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

for all $x, y \in X$.

THEOREM 2.2. *Let X and Y be vector spaces. If f is any solution of the functional equation (1.1), then f satisfies the functional equation (2.2). In other words, the general solution of the functional equation (1.1) is a quadratic mapping.*

Proof. We note that the functional equation (1.1) reduces to the functional equation (2.2) by putting $x_1 := x, x_2 := y$ and $x_3 = \cdots = x_n := 0$ in (1.1). \square

Now, we study the generalized Hyers–Ulam stability of the functional equation (1.1) by using direct method. As a matter of convenience, we use the following notation:

$$\begin{aligned} Df(x_1, \dots, x_n) &:= \sum_{i=1}^n \sum_{1 \leq i_1 \leq \dots \leq i_n \leq n} f\left(kx_i + \sum_{j=1, j \neq i}^{n-k} x_{i_j}\right) \\ &\quad - {}_{n-2}C_{n-k-1} \times (n+k-1) f\left(\sum_{i=1}^n x_i\right) \\ &\quad - {}_{n-2}C_{n-k-1} \times \frac{nk(k-1)}{n-k} \sum_{i=1}^n f(x_i) \end{aligned}$$

for all $x_1, \dots, x_n \in X$.

THEOREM 2.3. *Suppose X is a vector space and Y is a Banach space. Let $\varphi : X^n \rightarrow [0, \infty)$ be a function satisfying*

$$(2.3) \quad \Phi_1(x_1, \dots, x_n) := \sum_{j=0}^{\infty} \frac{\varphi(n^j x_1, \dots, n^j x_n)}{n^{2j}} < \infty$$

for all $x_1, \dots, x_n \in X$. If a mapping $f : X \rightarrow Y$ satisfies the functional equation

$$(2.4) \quad \|Df(x_1, \dots, x_n)\| \leq \varphi(x_1, \dots, x_n)$$

for all $x_1, \dots, x_n \in X$, then there exists a unique quadratic mapping $Q_1 : X \rightarrow Y$ such that

$$(2.5) \quad \|f(x) - Q_1(x)\| \leq \frac{M}{n^2} \Phi_1(x, \dots, x)$$

for all $x \in X$, where $M := \frac{n-k}{n-2C_{n-k-1} \times [k(k-1)]}$.

Proof. Letting $x_1 = \dots = x_n := x$ in (2.4), we obtain

$$\|f(nx) - n^2 f(x)\| \leq M \varphi(x, \dots, x)$$

for all $x \in X$. Dividing the above inequality by n^2 , one has

$$\|f(x) - \frac{f(nx)}{n^2}\| \leq \frac{M}{n^2} \varphi(x, \dots, x)$$

for all $x \in X$. Replacing x by $n^l x$ and dividing by n^{2l} , we get

$$\left\| \frac{f(n^l x)}{n^{2l}} - \frac{f(n^{l+1} x)}{n^{2(l+1)}} \right\| \leq \frac{M}{n^{2(l+1)}} \varphi(n^l x, \dots, n^l x)$$

for all $x \in X$ and all $l \in \mathbb{N} \cup \{0\}$. Thus, for any $p \in \mathbb{N}$, one deduces that

$$(2.6) \quad \left\| \frac{f(n^l x)}{n^{2l}} - \frac{f(n^{l+p} x)}{n^{2(l+p)}} \right\| \leq \frac{M}{n^2} \sum_{i=l}^{l+p-1} \frac{\varphi(n^i x, \dots, n^i x)}{n^{2i}}$$

for all $x \in X$ and all $l \in \mathbb{N} \cup \{0\}$. In view of (2.6), it is easily checked that the sequence $\{\frac{f(n^l x)}{n^{2l}}\}$ is Cauchy in Y . Since Y is complete, the sequence is convergent in Y . Hence, we may define a mapping $Q_1 : X \rightarrow Y$ by

$$(2.7) \quad Q_1(x) := \lim_{l \rightarrow \infty} \frac{f(n^l x)}{n^{2l}}$$

for all $x \in X$. Moreover, if we take $l = 0$ and $p \rightarrow \infty$ in (2.6), we arrive at the approximation (2.5). By the definition of Q_1 , we gain from (2.4)

and (2.7) that

$$\begin{aligned} \|DQ_1(x_1, \dots, x_n)\| &= \lim_{l \rightarrow \infty} \left\| \frac{Df(n^l x_1, \dots, n^l x_n)}{n^{2l}} \right\| \\ &\leq \lim_{l \rightarrow \infty} \frac{\varphi(n^l x_1, \dots, n^l x_n)}{n^{2l}} = 0 \end{aligned}$$

for all $x_1, \dots, x_n \in X$. Hence Q_1 satisfies the functional equation (1.1), and so it is quadratic.

To show that the uniqueness of Q_1 , we suppose that there exists another quadratic mapping $Q'_1 : X \rightarrow Y$ which satisfies the inequality

$$(2.8) \quad \|f(x) - Q'_1(x)\| \leq \frac{M}{n^2} \Phi_1(x, \dots, x)$$

for all $x \in X$, however $Q_1(y) \neq Q'_1(y)$ for some $y \in X$. Then there is a positive constant $\varepsilon > 0$ such that

$$0 < \varepsilon < \|Q_1(y) - Q'_1(y)\|.$$

For such given $\varepsilon > 0$, it follows from the assumption (2.3) that there is a positive integer $l_0 \in \mathbb{N}$ such that

$$\frac{2M}{n^2} \sum_{k=l_0}^{\infty} \frac{\varphi(n^k y, \dots, n^k y)}{n^{2k}} < \varepsilon.$$

Since Q_1 and Q'_1 are quadratic mappings, we see from (2.5) and (2.8) that

$$\begin{aligned} \varepsilon &< \|Q_1(y) - Q'_1(y)\| = \frac{1}{n^{2l_0}} \|Q_1(n^{l_0} y) - Q'_1(n^{l_0} y)\| \\ &\leq \frac{1}{n^{2l_0}} [\|Q_1(n^{l_0} y) - f(n^{l_0} y)\| + \|f(n^{l_0} y) - Q'_1(n^{l_0} y)\|] \\ &\leq \frac{2M}{n^{2(l_0+1)}} \sum_{i=0}^{\infty} \frac{\varphi(n^{i+l_0} y, \dots, n^{i+l_0} y)}{n^{2i}} = \frac{2M}{n^2} \sum_{k=l_0}^{\infty} \frac{\varphi(n^k y, \dots, n^k y)}{n^{2k}} \\ &< \varepsilon, \end{aligned}$$

which leads a contradiction. Hence Q_1 is a unique quadratic mapping satisfying (2.5). \square

THEOREM 2.4. *Let X be a vector space and Y a Banach space. Suppose there exists a function $\varphi : X^n \rightarrow [0, \infty)$ satisfying*

$$\Phi_2(x_1, \dots, x_n) := \sum_{i=1}^{\infty} n^{2i} \varphi\left(\frac{x_1}{n^i}, \dots, \frac{x_n}{n^i}\right) < \infty$$

for all $x_1, \dots, x_n \in X$. If a mapping $f : X \rightarrow Y$ satisfies the functional inequality (2.4), then there exists a unique quadratic mapping $Q_2 : X \rightarrow Y$ such that

$$\|f(x) - Q_2(x)\| \leq \frac{M}{n^2} \Phi_2(x, \dots, x)$$

for all $x \in X$.

Proof. We see from (2.4) that

$$\|f(nx) - n^2 f(x)\| \leq M \varphi(x, \dots, x)$$

for all $x \in X$. Thus, it follows that

$$\|f(x) - n^{2l} f\left(\frac{x}{n^l}\right)\| \leq \frac{M}{n^{2l}} \sum_{i=1}^n n^{2i} \varphi\left(\frac{x}{n^i}, \dots, \frac{x}{n^i}\right)$$

for all $x \in X$ and all $l \in \mathbb{N}$. Applying the same argument as in the proof of Theorem 2.3, we get the desired results. \square

Next, we recall one of the fundamental result of fixed point theory from [8].

PROPOSITION 2.5. *Let (X, d) be a generalized complete metric space and let $J : X \rightarrow X$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n , or there exists a positive integer n_0 such that

- (1) $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;
- (2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;
- (3) y^* is the unique fixed point of J in the set

$$Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\};$$

- (4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

Now, we will apply the fixed point method to investigate the generalized Hyers–Ulam stability of the functional equation (1.1).

THEOREM 2.6. *Let X be a vector space and Y a Banach space. Suppose there exists a positive number L_1 with $0 < L_1 < 1$ for which a function $\varphi : X^n \rightarrow [0, \infty)$ satisfies*

$$(2.9) \quad \varphi(nx_1, \dots, nx_n) \leq n^2 L_1 \varphi(x_1, \dots, x_n)$$

for all $x_1, \dots, x_n \in X$. If a mapping $f : X \rightarrow Y$ satisfies the functional inequality (2.4), then there exists a unique quadratic mapping $Q_1 : X \rightarrow Y$ such that

$$(2.10) \quad \|f(x) - Q_1(x)\| \leq \frac{M}{n^2(1-L_1)}\varphi(x, \dots, x)$$

for all $x \in X$.

Proof. First, we denote a set of mappings from X to Y by

$$Y^X := \{g : X \rightarrow Y \text{ a mapping}\},$$

and define a generalized metric d on Y^X as follows:

$$d(g, h) := \inf\{\alpha \in [0, \infty] : \|g(x) - h(x)\| \leq \alpha\varphi(x, \dots, x), \forall x \in X\}.$$

Then we may show that (Y^X, d) is a complete generalized metric space (see the proof of Theorem 2.1 of [6]). Now, we define a operator $J_1 : Y^X \rightarrow Y^X$ by

$$J_1g(x) := \frac{g(nx)}{n^2}$$

for all $g \in S$ and all $x \in X$. For any α with $d(g, h) < \alpha$, we can write

$$\begin{aligned} \|J_1g(x) - J_1h(x)\| &= \left\| \frac{g(nx)}{n^2} - \frac{h(nx)}{n^2} \right\| \\ &\leq \frac{1}{n^2}\alpha\varphi(nx, \dots, nx) \leq L_1\alpha\varphi(x, \dots, x), \end{aligned}$$

which yields $d(J_1g, J_1h) \leq L_1\alpha$ and so $d(J_1g, J_1h) \leq L_1d(g, h)$ by letting $\alpha \rightarrow d(g, h)^+$. So J_1 is a strictly contractive operator with Lipschitz constant L_1 on Y^X .

Second, if we set $x_1 = \dots = x_n := x$ in the hypothesis (2.4) and divide both sides by n^2 , then we have

$$\left\| f(x) - \frac{f(nx)}{n^2} \right\| \leq \frac{M}{n^2}\varphi(x, \dots, x)$$

for all $x \in X$, which implies

$$d(f, J_1f) \leq \frac{M}{n^2} < \infty.$$

Thus according to Proposition 2.5, there exists a mapping $Q_1 : X \rightarrow Y$, which is a fixed point of J_1 . Hence

$$Q_1(x) = J_1Q_1(x) = \frac{Q_1(nx)}{n^2}$$

and it satisfies

$$(2.11) \quad \lim_{k \rightarrow \infty} d(J_1^k f, Q_1) = 0, \text{ or } \lim_{k \rightarrow \infty} \frac{f(n^k x)}{n^{2k}} = Q_1(x)$$

for all $x \in X$. Using Proposition 2.5 once more, one also has

$$d(f, Q_1) \leq \frac{1}{1 - L_1} d(f, J_1 f) \leq \frac{M}{n^2(1 - L_1)},$$

and the mapping Q_1 is a unique fixed point of J_1 in the set

$$\Delta = \{g \in Y^X : d(J_1 f, g) < \infty\}.$$

This implies that Q_1 is a unique mapping satisfying

$$\|f(x) - Q_1(x)\| \leq \frac{M}{n^2(1 - L_1)} \varphi(x, \dots, x)$$

for all $x \in X$.

Finally, we prove that the mapping Q_1 is quadratic. It follows from (2.4) and (2.11) that

$$\begin{aligned} \|DQ_1(x_1, \dots, x_n)\| &= \lim_{k \rightarrow \infty} \frac{1}{n^{2k}} \|Df(n^k x_1, \dots, n^k x_n)\| \\ &\leq \lim_{k \rightarrow \infty} \frac{1}{n^{2k}} \varphi(n^k x_1, \dots, n^k x_n) \\ &\leq \lim_{k \rightarrow \infty} L_1^k \varphi(x_1, \dots, x_n) = 0 \end{aligned}$$

for all $x_1, \dots, x_n \in X$. Hence Q_1 satisfies the functional equation (1.1) and so the mapping $Q_1 : X \rightarrow Y$ is quadratic, as desired. \square

THEOREM 2.7. *Let X be a vector space and Y a Banach space. Assume that $\varphi : X^n \rightarrow [0, \infty)$ is a function satisfying*

$$\varphi\left(\frac{x_1}{n}, \dots, \frac{x_n}{n}\right) \leq \frac{L_2}{n^2} \varphi(x_1, \dots, x_n)$$

for some real number L_2 with $0 < L_2 < 1$ and all $x_1, \dots, x_n \in X$. If a mapping $f : X \rightarrow Y$ satisfies the functional inequality (2.4), then there exists a unique quadratic mapping $Q_2 : X \rightarrow Y$ such that

$$\|f(x) - Q_2(x)\| \leq \frac{M}{(1 - L_2)} \varphi\left(\frac{x}{n}, \dots, \frac{x}{n}\right)$$

for all $x \in X$.

Proof. Let

$$Y^X := \{g : X \rightarrow Y \text{ a mapping}\},$$

and define a generalized metric d on Y^X by

$$d(g, h) := \inf\{\alpha \in [0, \infty] : \|g(x) - h(x)\| \leq \alpha\varphi\left(\frac{x}{n}, \dots, \frac{x}{n}\right), \forall x \in X\}.$$

Then (Y^X, d) is a complete generalized metric space. On the space Y^X , we define a operator $J_2 : Y^X \rightarrow Y^X$ by

$$J_2g(x) := n^2g\left(\frac{x}{n}\right)$$

for all $g \in Y^X$ and all $x \in X$, then we see from (2.4) that

$$\|f(x) - n^2f\left(\frac{x}{n}\right)\| \leq M\varphi\left(\frac{x}{n}, \dots, \frac{x}{n}\right)$$

for all $x \in X$, which induces $d(f, J_2f) \leq M$ by definition.

The rest of proof follows from the similar argument to the corresponding part of Theorem 2.6. \square

COROLLARY 2.8. *Let X be a normed space and Y a Banach space. If a mapping $f : X \rightarrow Y$ satisfies the functional inequality*

$$\|Df(x_1, \dots, x_n)\| \leq \theta\left(\sum_{i=1}^n \|x_i\|^r\right)$$

for all $x_1, \dots, x_n \in X$, where $0 < r \neq 2$, then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{Mn\theta}{|n^2 - n^r|} \|x\|^r$$

for all $x \in X$.

COROLLARY 2.9. *Let X be a normed space and Y a Banach space, and let r_1, \dots, r_n be real numbers with $0 < \sum_{i=1}^n r_i := r \neq 2$. If a mapping $f : X \rightarrow Y$ satisfies the functional inequality*

$$\|Df(x_1, \dots, x_n)\| \leq \theta\left(\prod_{i=1}^n \|x_i\|^{r_i}\right)$$

for all $x_1, \dots, x_n \in X$, then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{M\theta}{|n^2 - n^r|} \|x\|^r$$

for all $x \in X$.

COROLLARY 2.10. *Let X be a vector space and Y a Banach space. Let $\varepsilon \geq 0$ be any given real number. Suppose that a mapping $f : X \rightarrow Y$ satisfies the functional inequality*

$$\|Df(x_1, \dots, x_n)\| \leq \varepsilon$$

for all $x_1, \dots, x_n \in X$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{M\varepsilon}{(n^2 - 1)}$$

for all $x \in X$.

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